Intro to Formal Political Analysis: Strategic Games and Nash Equilibria in Pure Strategies

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What is a “strategic game”?

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Osborne, Definition 13.1

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- a set of **players**
- for each player, a set of **actions**

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- for each player, **preferences** over the set of action profiles

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This is also called a “normal-form game”

Example: The Prisoner’s Dilemma

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- If only one of them talks, that prisoner will get a plea deal and serve no time while the other will be convicted of the major crime

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- So if neither talks, they both get convicted of the minor crime but neither gets convicted of the major crime
- If only one of them talks, that prisoner will get a plea deal and serve no time while the other will be convicted of the major crime
- If both talk, they are both convicted of the major crime, but get a slightly reduced sentence for confessing

The Prisoner’s Dilemma as a normal form game

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    - (C, C); both are only convicted of the minor crime only

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    - (D, D); both are convicted of the felony with a reduced sentence

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    - (C, D); 1 is convicted of the major crime and pays full freight

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    - (C, C); both are only convicted of the minor crime only
    - (D, D); both are convicted of the felony with a reduced sentence
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  - Player 2's ordering is (C, D), (C, C), (D, D), and (D, C).

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    - (C, D); 1 is convicted of the major crime and pays full freight
  - Player 2's ordering is (C, D), (C, C), (D, D), and (D, C).

A utility function that represents Player 1's preferences is

$$u_1(D, C) = 3,\ u_1(C, C) = 2,\ u_1(D, D) = 1,\ u_1(C, D) = 0$$

Matrix representation of The Prisoner’s Dilemma

Matrix representation of The Prisoner's Dilemma

	C	D
C	2, 2	0, 3
D	3, 0	1, 1

Matrix representation of The Prisoner's Dilemma

I produced that for these slides using R Markdown with the code:

````
```{r prisoners-dilemma-matrix}
library(kableExtra)
game = data.frame(
    C = c("$2, 2$", "$3, 0$"),
    D = c("$0, 3$", "$1, 1$"),
    row.names = c("C", "D")
)
kable(game, escape = FALSE, align = "c") %>%
    kable_styling(position = "center")
```
````

Matrix representation of The Prisoner's Dilemma

You could produce a similar payoff matrix in LaTeX with the code:

```
\begin{game}{2}{2}
    & $C$       & $D$       \\
$C$ & $2, 2$    & $0, 3$    \\
$D$ & $3, 0$    & $1, 1$
\end{game}
```

(this requires you add `\usepackage{sgame}` to your preamble)

Nash Equilibrium

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Osborne, Definition 23.1

The action profile a^* in a strategic game... is a **Nash equilibrium** if, for every player i and every action a_i of player i, a^* is at least as good according to player i's preferences as the action profile (a_i, a^*_{-i}) in which player i chooses a_i while every other player j chooses a^*_j. Equivalently, for every player i,

$$u_i\left(a^*\right) \geq u_i\left(a_i, a^*_{-i}\right) \text{ for every action } a_i \text{ of player } i,$$

where u_i is a payoff function that represents player i's preferences.

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$$u_i\left(a^*\right) \geq u_i\left(a_i, a^*_{-i}\right) \text{ for every action } a_i \text{ of player } i,$$

where u_i is a payoff function that represents player i's preferences.

In other words, a Nash equilibrium is an action profile where no player can gain by unilateral deviation

Nash equilibrium in The Prisoner's Dilemma

	C	D
C	2, 2	0, 3
D	3, 0	1, 1

Nash equilibrium in The Prisoner's Dilemma

	C	D
C	2, 2	0, 3
D	**3**, 0	1, 1

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- (D, D) is the only profile where neither can gain by deviating

Nash equilibrium in The Prisoner's Dilemma

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- (D, D) is the only profile where neither can gain by deviating
- It is therefore the unique Nash equilibrium

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	C	D
C	**2, 2**	0, **3**
D	**3**, 0	**1**, **1**

- (D, D) is the only profile where neither can gain by deviating
- It is therefore the unique Nash equilibrium
- Notice they'd both be better off in the action profile (C, C)

Nash equilibrium in The Prisoner's Dilemma

	C	D
C	**2, 2**	0, **3**
D	**3**, 0	**1**, **1**

- (D, D) is the only profile where neither can gain by deviating
- It is therefore the unique Nash equilibrium
- Notice they'd both be better off in the action profile (C, C)
- But Nash equilibrium is about *unilateral* deviation

Domination

Domination

Osborne, Definition 45.1

In a strategic game... player i's action a_i'' **strictly dominates** her action a_i' if

$$u_i\left(a_i'', a_{-i}\right) > u_i\left(a_i', a_{-i}\right) \text{ for every list } a_{-i} \text{ of the other players' actions,}$$

where u_i is a payoff function that represents player i's preferences. We say that the action a_i' is **strictly dominated**.

Domination

Osborne, Definition 46.1

In a strategic game... player i's action a_i'' **weakly dominates** her action a_i' if

$$u_i\left(a_i'', a_{-i}\right) \geq u_i\left(a_i', a_{-i}\right) \text{ for every list } a_{-i} \text{ of the other players' actions,}$$

and

$$u_i\left(a_i'', a_{-i}\right) > u_i\left(a_i', a_{-i}\right) \text{ for some list } a_{-i} \text{ of the other players' actions,}$$

where u_i is a payoff function that represents player i's preferences. We say that the action a_i' is **weakly dominated**.

Domination

In other words,

1. Strict domination: Action A strictly dominates action B if the player strictly prefers all action profiles where they play A to all action profiles where they play B
2. Weak domination: A weakly dominates B if we change "strictly prefers" to "weakly prefers" **and** require the preference is strict in at least one action profile

Domination in The Prisoner's Dilemma

	C	D
C	2, 2	0, 3
D	3, 0	1, 1

Domination in The Prisoner's Dilemma

	C	D
C	2, 2	0, 3
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Domination in The Prisoner's Dilemma

	C	D
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Domination in The Prisoner’s Dilemma

	C	D
C	2, 2	0, **3**
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Domination in The Prisoner's Dilemma

	C	D
C	2, 2	0, **3**
D	**3**, 0	**1**, **1**

Domination in The Prisoner's Dilemma

	C	D
C	2, 2	0, **3**
D	**3**, 0	**1**, **1**

Whether Player 1 plays *C* or *D*, Player 2 strictly prefers to play *D*. Therefore, for Player 2, *D* **strictly dominates** *C*.

Another example of domination

	L	R
T	1	0
M	2	1
B	3	2

Another example of domination

	L	R
T	**1**	**0**
M	**2**	**1**
B	3	2

Another example of domination

	L	R
T	**1**	**0**
M	**2**	**1**
B	3	2

Another example of domination

	L	R
T	1	0
M	**2**	**1**
B	**3**	**2**

Another example of domination

	L	R
T	1	0
M	**2**	**1**
B	**3**	**2**

Another example of domination

	L	R
T	1	0
M	2	1
B	3	2

For Player 1, *M* **strictly dominates** *T*, and *B* **strictly dominates** *M*.

Example: Bach or Stravinski?

Two people want to go out on Friday. There’s a concert playing music by Bach and a concert playing music by Stravinski. One person prefers Bach while the other prefers Stravinski. For both of them, the worst option is going to a concert alone.

Example: Bach or Stravinski?

- Players: The two people

Example: Bach or Stravinski?

- Players: The two people
- Actions: Each player can go to the Bach concert (B) or the Stravinski concert (S)

Example: Bach or Stravinski?

- Players: The two people
- Actions: Each player can go to the Bach concert (*B*) or the Stravinski concert (*S*)
- Preferences:

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- Actions: Each player can go to the Bach concert (B) or the Stravinski concert (S)
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  - From best to worst, Player 1 prefers $(B, B), (S, S), (B, S), (S, B)$

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  - From best to worst, Player 2 prefers $(S, S), (B, B), (B, S), (S, B)$

Example: Bach or Stravinski?

- Players: The two people
- Actions: Each player can go to the Bach concert (B) or the Stravinski concert (S)
- Preferences:
  - From best to worst, Player 1 prefers $(B, B), (S, S), (B, S), (S, B)$
  - From best to worst, Player 2 prefers $(S, S), (B, B), (B, S), (S, B)$
  - Assuming they don't care about which concert they go to if alone, a utility function that represents Player 1's preferences is

$$u_1(B, B) = 2, \; u_1(S, S) = 1, \; u_1(B, S) = u_1(S, B) = 0$$

Example: Bach or Stravinski?

	B	S
B	2, 1	0, 0
S	0, 0	1, 2

Example: Bach or Stravinski?

	B	S
B	**2**, 1	0, 0
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Example: Bach or Stravinski?

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B	**2**, 1	0, 0
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	B	S
B	**2**, **1**	0, 0
S	0, 0	**1**, 2

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	B	S
B	**2**, **1**	0, 0
S	0, 0	**1**, **2**

Example: Bach or Stravinski?

	B	S
B	**2**, **1**	0, 0
S	0, 0	**1**, **2**

In both the (B, B) and (S, S) profiles, neither player can gain from unilateral deviation.

Therefore, both action profiles are a Nash equilibrium.

Example: Bach or Stravinski?

	B	S
B	**2**, **1**	0, 0
S	0, 0	**1**, **2**

In both the (B, B) and (S, S) profiles, neither player can gain from unilateral deviation.

Therefore, both action profiles are a Nash equilibrium.

Does any of Player 1's actions weakly or strictly dominate any of their other actions? How about for Player 2?

Example: Stag Hunt

Osborne, page 20 in the 9th printing

A sentence in *Discourse on the origin and foundations of inequality among men* (1755) by the philosopher Jean-Jacques Rousseau discusses a group of hunters who wish to catch a stag… They will succeed if they all remain sufficiently attentive, but each is tempted to desert her post and catch a hare.

Example: Stag Hunt

- Players: The hunters

Example: Stag Hunt

- Players: The hunters
- Actions: Each can remain attentive to the stag (*S*) or desert and catch a hare (*H*)

Example: Stag Hunt

- Players: The hunters
- Actions: Each can remain attentive to the stag (*S*) or desert and catch a hare (*H*)
- Preferences: Each best prefers a profile where every player chooses *S*, next prefers any profile where they personally choose *H*, and lastly prefer any profile where they choose *S* and at least one other player chooses *H*

Example: Stag Hunt

A matrix representation of the two-person stag hunt is

	S	H
S	2, 2	0, 1
H	1, 0	1, 1

Example: Stag Hunt

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A matrix representation of the two-person stag hunt is

	S	H
S	**2**, **2**	0, 1
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In both the (S, S) and (H, H) profiles, neither player can gain from unilateral deviation.

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Example: Matching Pennies

Two people simultaneously show each other a side of a penny. If they show each other the same side, Player 2 pays Player 1 \$1, while if they show each other different sides, Player 1 pays Player 2 \$1.

Example: Matching Pennies

- Players: The two people

Example: Matching Pennies

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  - From best to worst, Player 1 prefers (H, H) or (T, T), then (H, T) or (T, H)

Example: Matching Pennies

- Players: The two people
- Actions: Each player can show heads (H) or show tails (T)
- Preferences:
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Example: Matching Pennies

- Players: The two people
- Actions: Each player can show heads (H) or show tails (T)
- Preferences:
  - From best to worst, Player 1 prefers (H, H) or (T, T), then (H, T) or (T, H)
  - From best to worst, Player 2 prefers (H, T) or (T, H), then (H, H) or (T, T)
  - We can represent their preferences by how much money they gain or lose;

$$u_1(H, H) = u_1(T, T) = 1, \;\; u_1(H, T) = u_1(T, H) = -1$$

Example: Matching Pennies

	H	T
H	1, -1	-1, 1
T	-1, 1	1, -1

Example: Matching Pennies

	H	T
H	**1**, -1	-1, 1
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Example: Matching Pennies

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There is no pure strategy Nash equilibrium in the Matching Pennies game.

Example: Matching Pennies

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Example: Voting

Consider the following scenario:

- Two candidates, A and B, run for office

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- Two candidates, A and B, run for office
- There are N voters, with N odd; some of them (a majority) prefer candidate A to candidate B, and the rest prefer B to A.

Example: Voting

Consider the following scenario:

- Two candidates, A and B, run for office
- There are N voters, with N odd; some of them (a majority) prefer candidate A to candidate B, and the rest prefer B to A.
- There is an election with mandatory voting; the candidate with the most votes wins

Example: Voting

- Players: The N voters

Example: Voting

- Players: The N voters
- Actions: Each can vote for A or vote for B

Example: Voting

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Example: Voting

- Players: The N voters
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- Preferences:
  - All players are indifferent between action profiles where A wins

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Example: Voting

- Players: The N voters
- Actions: Each can vote for A or vote for B
- Preferences:
  - All players are indifferent between action profiles where A wins
  - All players are indifferent between action profiles where B wins
  - Players who prefer candidate A prefer any action profile where A wins to any action profile where B wins, and vice versa for voters who prefer candidate B

Example: Voting

Questions:

- Do the players have an action that dominates their other action?

Example: Voting

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- Do the players have an action that dominates their other action?
- Is there any Nash equilibrium where candidate A wins the election?

Example: Voting

Questions:

- Do the players have an action that dominates their other action?
- Is there any Nash equilibrium where candidate A wins the election?
- Is there any Nash equilibrium where candidate B wins the election?

Dominance Solvable Games

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  - Order of elimination does **not** matter

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- If you are also eliminating weakly dominated strategies, the remaining action profile will be a Nash equilibrium, but you may have eliminated other Nash equilibria (IEWDS / IDWDS)

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- A game is **dominance solvable** if iterated elimination of dominated actions leaves only one actions left for each player
- If you are eliminating only strictly dominated strategies, the remaining action profile is the unique Nash equilibrium to the game (IESDS / IDSDS)
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- If you are also eliminating weakly dominated strategies, the remaining action profile will be a Nash equilibrium, but you may have eliminated other Nash equilibria (IEWDS / IDWDS)
    - Order of elimination **does** matter

Dominance Solvable Games: Examples

The Prisoner's Dilemma

	C	D
C	2, 2	0, 3
D	3, 0	1, 1

Dominance Solvable Games: Examples

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Dominance Solvable Games: Examples

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Dominance Solvable Games: Examples

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Dominance Solvable Games: Examples

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Dominance Solvable Games: Examples

Bach or Stravinski

	B	S
B	2, 1	0, 0
S	0, 0	1, 2

Dominance Solvable Games: Examples

	L	C	R
T	0, 0	1, 0	1, 1
M	1, 0	1, 1	3, 2
B	1, 0	2, 1	2, 2

Dominance Solvable Games: Examples

	L	C	R
T	0, 0	1, 0	1, 1
M	1, 1	1, 1	3, 2
B	1, 1	2, 1	2, 2

More Voting: The Hotelling-Downs Model

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Consider the following model of elections:

- Two candidates seek election & *only* care about winning office

More Voting: The Hotelling-Downs Model

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- Suppose there is a unidimensional policy space
- The candidates run for office on a platform (a point)
- There are a continuum of voters each with an ideal point
- They vote for the candidate closest to their ideal point
- (If multiple platforms are equidistant from a point, the candidates evenly split the relevant voters)

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Questions:

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- Are you already familiar with the game's solution?
- How would you find the solution to this game?

Even More Voting! The Calvert-Wittman Model

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- (These are sometimes called “citizen-candidates”)
- Does this change the Nash equilibrium? (Focus on the case where the two candidates’ ideal points are on opposite sides of the median)

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- What about asymmetric voter preferences?
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- What about multiple dimensions?
- Etc, etc, etc

Best Response Functions

The **best response function** of player *i* is

$$B_i\left(a_{-i}\right) = \left\{a_i \in A_i : u_i\left(a_i, a_{-i}\right) \geq u_i\left(a_i', a_{-i}\right) \forall\, a_i' \in A_i\right\}$$

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Osborne Proposition 36.1

The action profile a^* is a Nash equilibrium of a strategic game… if and only if every player's action is a best response to the other player's actions:

$$a_i^* \text{ is in } B_i\left(a_{-i}^*\right) \text{ for every player } i.$$

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$$B_1(x_2) = \begin{cases} \{x_1 : x_2 < x_1 < 2m - x_2\} & \text{if } x_2 < m \\ \{m\} & \text{if } x_2 = m \\ \{x_1 : 2m - x_2 < x_1 < x_2\} & \text{if } x_2 < m \end{cases}$$

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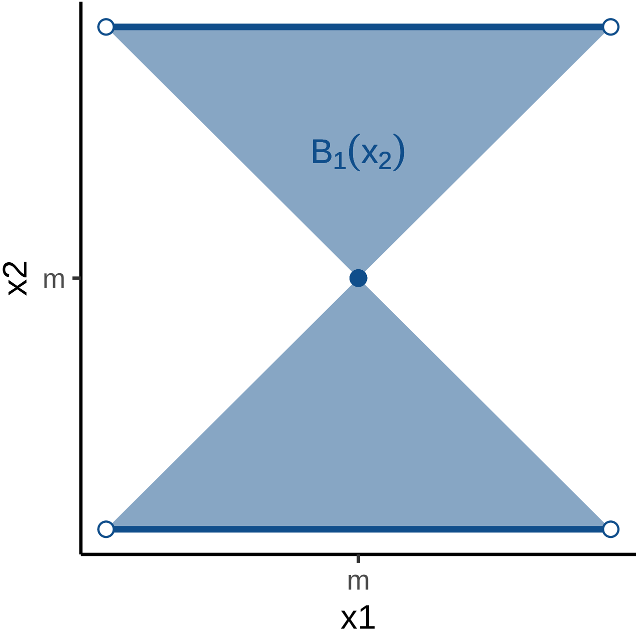

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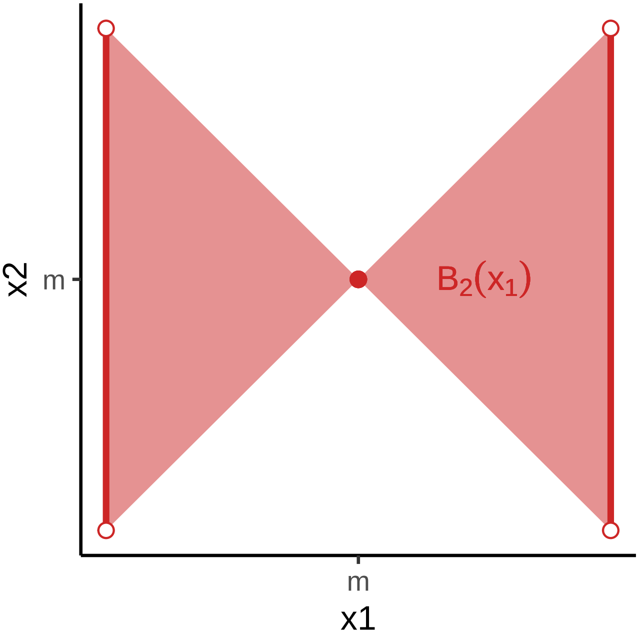

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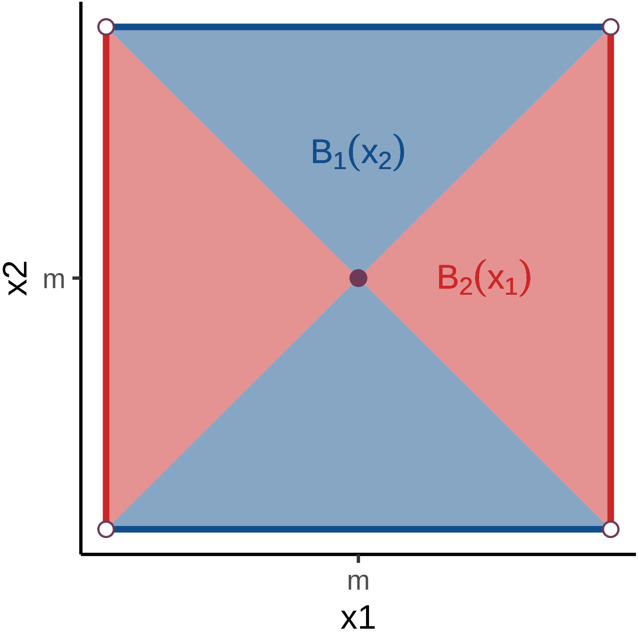

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$$
\begin{aligned}
u_i(c_1, c_2) &= c_1 + c_2 + w - c_i + (w - c_i)(c_1 + c_2) \\
&= w + c_j + (w - c_i)(c_1 + c_2)
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$$u_i(c_1, c_2) = w + c_j + (w - c_i)(c_1 + c_2)$$

- How would you find the players' best response functions?

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u_i\left(c_i, c_j\right) &= w + c_j + (w - c_i)(c_i + c_j) \\
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\frac{\partial}{\partial c_i} u_i &= \frac{\partial}{\partial c_i} wc_i - \frac{\partial}{\partial c_i} c_i c_j - \frac{\partial}{\partial c_i} c_i^2 \\
&= w - c_j - 2c_i
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&= w - c_j - 2c_i \\
0 &\equiv w - c_j - 2c_i \\
2c_i &= w - c_j \\
c_i^* &= \frac{1}{2}\left(w - c_j\right)
\end{aligned}
$$

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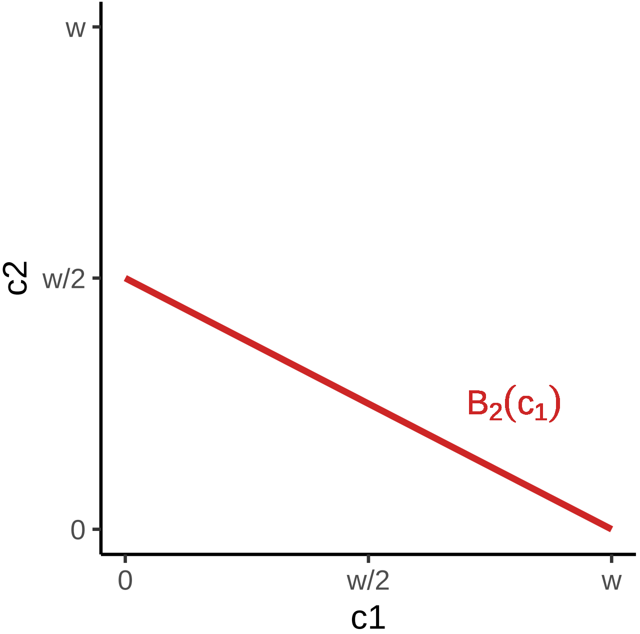

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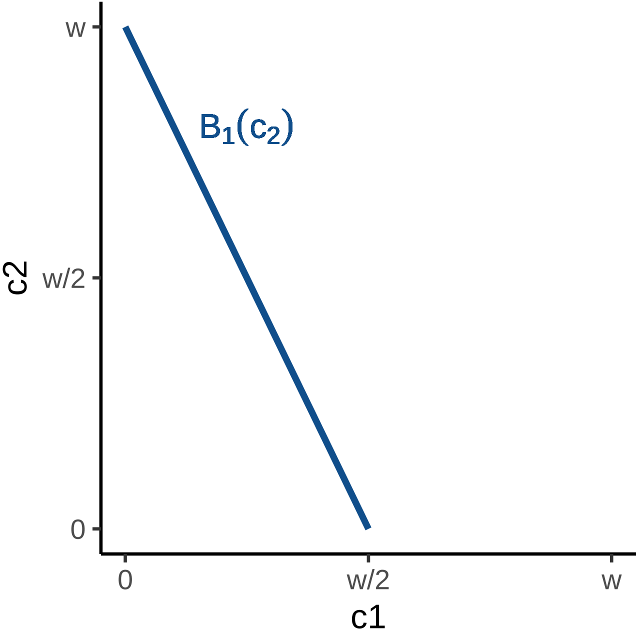

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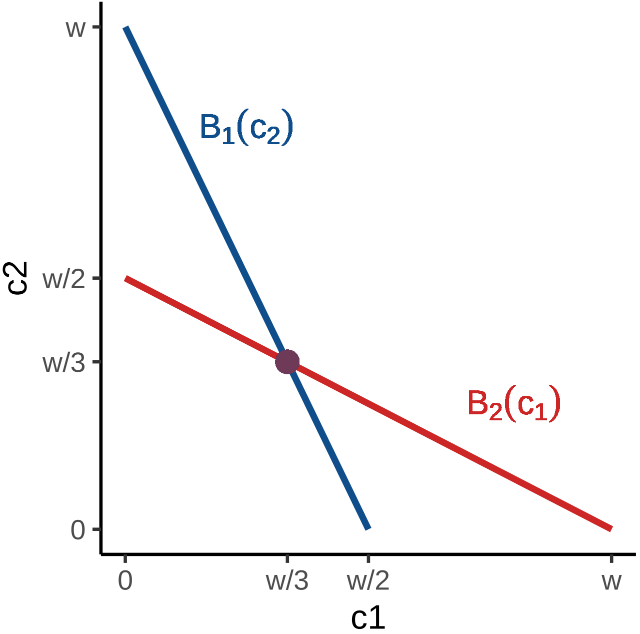

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$$c_i^* = \frac{1}{2}\left(w - c_j^*\right)$$

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c_i^* &= \frac{1}{2}\left(w - c_j^*\right) \\
c_i^* &= \frac{1}{2}\left(w - \frac{1}{2}\left(w - c_i^*\right)\right) \\
c_i^* &= w/4 + c_i^*/4 \\
4c_i^* &= w + c_i^* \\
3c_i^* &= w \\
c_i^* &= w/3
\end{aligned}
$$